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Some Results on the CR property of non-E-overlapping and depth-preserving TRS's

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Abstract

A term rewriting system (TRS) is said to be depth-preserving if for any rewrite rule and any variable appering in the both sides, the maximal depth of the variable occurences in left-hand-side is greater than or equal to that of the variable occurrences in the right-hand-side, and to be strongly depth-preserving if it is depth-preserving and for any rewrite rule and any variable appering in the left-hand-side, all the depths of the variable occurrences in the left-hand-side are the same. This paper shows that there exists non-E-overlapping and depth-preserving TRS's which do not satisfy the Church-Rosser property, but all the non-E-overlapping and strongly depth-preserving TRS's satisfy the Church-Rosser property.

1 Introduction

A term-rewriting system (TRS) is a set of directed equations (called rewrite rules). A TRS is Church-Rosser (CR) if any two interconvertible terms reduce to some common term by applications of the rewrite rules. Church-Rosser is an important property in various applications of TRS's and has received much attention so far [1-5,8-15]. Although the CR property is undecidable for general TRS's, many sufficient conditions for ensuring this property have been obtained [1,3,5,8-15]. For example, for noetherian (i.e. terminating) TRS's, the CR property is decidable and reduces to joinability of the critical pairs [5], and for nonterminating and linear TRS's, some sufficient conditions (e.g., nonoverlapping) have been given [3, 11].

On the other hand, for nonlinear and nonterminating TRS's, only a few results on the CR property have been obtained. Our previous paper [9,10,13] may be pioneer ones which have first given nontrivial conditions for the CR property. In [10], it was shown that if TRS's are non-E-overlapping (stronger than nonoverlapping) and right-ground, then they are CR. Here, a TRS is right-ground if no variables occur in the right-hand-side of a rewrite rule. This result is compared with an example given by G.Huet [3], i.e., a nonoverlapping, right-ground and non-CR TRS with the three rules: $f(x,x) \to a, f(x,g(x)) \to b, c \to g(c)$. Here, f,g,a,b,c are function symbols and x is a variable. The above result was extended in [9,13,14,15] and it was shown that if TRS's are non-E-overlapping and simple-right-linear, then they are CR. Here, a TRS is simple-right-linear if for any rewrite rule, the right-hand-side is linear (i.e., any variable occurs at most once in the term) and no variables occuring more than once in the left-hand-side occur in the right-hand-side. Moreover, it was shown that even if simple-right-linear TRS's are E-overlapping, some additional conditions ensure that they are CR [9,13,15].

However, these results were restricted to those on the CR property of subclasses of right-linear TRS's. On the other hand, if we omit the right-linearity condition, then it has been shown that only the non-E-overlapping condition is insufficient for ensuring the CR property of TRS's. For example, the following non-E-overlapping TRS R_1 is not CR: $R_1 = \{f(x,x) \rightarrow a, g(x) \rightarrow f(x,g(x)), c \rightarrow g(c)\}$ given by Barendregt and Klop. Here, f, g, a, c are function symbols and x is a variable.

In this paper, we consider the CR property of nonlinear, nonterminating and depth-preserving TRS's. Here, a TRS is depth-preserving if for each rule $\alpha \rightarrow \beta$ and any variable x appearing in both α and β,the maximal depth of the x ocurrences in α is greater than or equal to that of the x occurrences in β([6]). For example, TRS $R_2 = \{f(x, g(x)) \rightarrow h(k(x), x)\}$, where x is a variable, is depth-preserving, since the maximal depths of the x occurrences of the left-hand-side and of the right-hand-side are 2 and 2, respectively.

We first show that only the non-E-overlapping and depth-preserving properties are insufficient for ensuring the CR property. That is, the following TRS R_3 is not CR: $R_3 = \{f(x,x) \rightarrow a, c \rightarrow h(c, g(c)), h(x, g(x)) \rightarrow f(x, h(x, g(c)))\}$ where x is a variable. Note that R_3 is non-E-overlapping and depth-preserving,but R_3 is not CR, since $c \rightarrow h(c, g(c)) \rightarrow^* a$ and $c \rightarrow^* h(a, g(a))$, but a and $h(a, g(a))$ are not joinable. Note that R_3 is also non-duplicating, since for each rule the number of x occurrences of the left-hand side $\geq$ that of the right-hand side. Thus, non-E-overlapping, non-duplicating and depth-preserving conditions do not necessarily ensure CR.

Next, we introduce the notion of strongly depth-preserving property (stronger than the depth-preserving one). A TRS R is strongly depth-preserving if R is depth-preserving and for each $\alpha \rightarrow \beta$ and for any variable x appearing in α, all the depths of the x occurrences in α are the same. For example, TRS $R_4 = \{h(g(x), g(x)) \rightarrow f(x, h(x, g(c)))\}$ is strongly depth-preserving, since R_4 is depth-preserving and all the depths of x occurrences of the left-hand side are 2.

In this paper, we prove that non-E-overlapping and strongly depth-preserving TRS's are CR. For example, the following three TRS's R'_1, R'_3 and R_5 are ensured to be CR:

$$\begin{aligned}
R'_1 &= \{f(x,x) \rightarrow a,\ c \rightarrow g(c),\ g(x) \rightarrow f(x,x)\} \\
R'_3 &= \{f(x,x) \rightarrow a,\ c \rightarrow h(c, g(c)),\ h(g(x), g(x)) \rightarrow f(x, h(x, g(c)))\} \\
R_5 &= \{f(x,x) \rightarrow h(x,x,x)\}
\end{aligned}$$

This paper is organized as follows. Section 2 is devoted to definitions. In Section 3, we explain how to prove the above main theorem. In Section 4, we make concluding remarks about the strongly depth-preserving property.

2 Definitions

The following definitions and notations are similar to those in [3, 10]. Let X be a set of variables, F be a finite set of operation symbols and T be the set of terms constructed from X and F.

Definitions of $< O(M), M/u, M[u \leftarrow N], V(M), O_x(M) >$

For a term M, we use $O(M)$ to denote the set of occurrences (positions) of M, and M/u to denote the subterm of M at occurrence u, and $M[u \leftarrow N]$ to denote the term obtained form M by replacing the subterm M/u by term N, $V(M)$ to denote the set of variables in M, $O_x(M)$ to denote the set of occurrences of variable $x \in V(M)$.

Definitions of $< \bar{O}(M) >$

$\bar{O}(M)$ is the set of non-variable occurences, i.e.,
$\bar{O}(M) = O(M) - \cup_{x \in V(M)} O_x(M)$

Definition of $<$ $h(M)$ — height of M $>$

For a term M, $h(M) = Max\{|u| \mid u \in O(M)\}$. $h(M)$ is called "height of M".
Example.
$h(f(g(x))) = 2,\ h(a) = 0,\ h(g(x)) = 1.$

Definition of $<$ TRS $>$

A term-rewriting system (TRS) is a set of directed equations (called rewrite rules).

Definition of $<$ depth-preserving TRS R $>$

TRS R is depth-preserving
if $\forall \alpha \to \beta \in R\, \forall x \in V(\alpha) \quad Max\{|v| \mid v \in O_x(\beta)\} \leq Max\{|u| \mid u \in O_x(\alpha)\}$

Note

TRS R is depth-preserving if and only if R is locally increasing, i.e., $\exists l \geq 0$ such that $\forall \alpha \to \beta \in R$ $\forall \sigma : X \to T$, if $h(\sigma(\alpha)) < h(\sigma(\beta))$ then $h(\sigma(\alpha)) \leq l$

Definition of $<$ strongly depth-preserving TRS R $>$

TRS R is strongly depth-preserving
if R is depth-preserving and satisfies that $\forall \alpha \to \beta \in R\ \forall x \in V(\alpha)\ \forall u, v \in O_x(\alpha)$ $|u| = |v|$ hold.

Definition of $<$ parallel-one-step $\mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow}$ $>$

$$
\begin{array}{lll}
M \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} N & \text{iff} & \exists U \subseteq O(M) \text{ s.t.} \\
 & & \forall u, v \in U \quad u \neq v \Rightarrow u|v \text{ (disjoint)} \\
 & & \forall u \in U \quad M/u \Leftrightarrow N/u \\
 & & \quad N = M[u \leftarrow N/u, u \in U]
\end{array}
$$

where $M/u \Leftrightarrow N/u$ is one step reduction between $\{M/u, N/u\} = \{\sigma(\alpha), \sigma(\beta)\}$ for some $\alpha \to \beta \in R$ and $\sigma : X \to T$.
In this case, let $R(M \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} N) = U$.
(Note. $U = \phi$ is allowed.)

Example.
Let $R = \{a \to c\}$, then $f(c, g(a)) \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} f(a, g(c))$.

We assume that $\gamma : M_0 \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} M_1 \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} \cdots \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} M_n$ in the following definitions.

Definition of $<$ $R(\gamma)$, $MR(\gamma)$, u-invariant $>$

$R(\gamma) = \{u_i \mid u_i \in R(M_i \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} M_{i+1})\ (0 \leq i \leq n)\}$
$MR(\gamma)$ is the set of minimal occurrences in $R(\gamma)$.
For $u \in O(M_0)$, if there exists no $v \in R(\gamma)$ such that $v \leq u$, then γ is said to be u-invariant.

Definition of $<$ composition, cut of reduction sequence $>$

Let $\delta : N_0 \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} N_1 \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} \cdots \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} N_k$. If $M_n = N_0$, then the composition of γ and δ, i.e., $M_0 \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} M_1 \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} \cdots \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} M_n (= N_0) \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} N_1 \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} \cdots \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} N_k$ is denoted by $(\gamma; \delta)$.

Let γ be u-invariant, then the cut sequence of γ at u is
$\gamma/u = (M_0/u \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} M_1/u \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} \cdots \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} M_n/u)$.

Definition of < $H(\gamma)$ — the height of reduction sequence >

$H(\gamma) = Max\{h(M_i) \mid 0 \leq i \leq n\}$

Example.

Let $\gamma : f(c) \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} f(g(c)) \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} a$, then $H(\gamma) = h(f(g(c))) = 2$.

Definition of < $|\gamma|_p$ — the number of parallel reduction steps of γ >

$|\gamma|_p = n$

Note.

If $\delta : M \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} M$, then $|\delta|_p = 1$.

Example.

Let $\gamma : f(c) \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} f(g(c)) \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} a$, then $|\gamma|_p = 2$.

Definition of < $net(\gamma)$ >

$net(\gamma)$ is the sequence obtained from γ by removing all $M_i \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} M_{i+1}$ satisfying $M_i = M_{i+1}$, $0 \leq i < n$.

Example.

Let $\gamma : f(c) \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} f(g(c)) \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} a \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} a$, then $net(\gamma) : f(c) \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} f(g(c)) \mathrel{\leftarrow\!\!\!\!+\!\!\!\!\rightarrow} a$.

Definition of < $|\gamma|_{np}$ >

$|\gamma|_{np} = |net(\gamma)|_p$

Definitions of < $left(\gamma, h), right(\gamma, h), width(\gamma, h), ldis(\gamma, h), rdis(\gamma, h)$ >

$left(\gamma, h)$	$=$	$Min\{i \mid h(M_i) = h\}$	if $\exists i \ (0 \leq i \leq n)$ s.t. $h(M_i) = h$ and $\forall j (0 \leq j < i) \ h(M_j) < h$
	$=$	$\perp$	otherwise
$right(\gamma, h)$	$=$	$Max\{i \mid h(M_i) = h\}$	if $\exists i \ (0 \leq i \leq n)$ s.t. $h(M_i) = h$ and $\forall j \ (i < j \leq n) \ h(M_j) < h$
	$=$	$\perp$	otherwise
$left(\gamma, h) \downarrow$	$\stackrel{def}{=}$	$left(\gamma, h) \neq \perp$	
$right(\gamma, h) \downarrow$	$\stackrel{def}{=}$	$right(\gamma, h) \neq \perp$	
$left(\gamma, h) \uparrow$	$\stackrel{def}{=}$	$left(\gamma, h) = \perp$	
$right(\gamma, h) \uparrow$	$\stackrel{def}{=}$	$right(\gamma, h) = \perp$	
$width(\gamma, h)$	$=$	$right(\gamma, h) - left(\gamma, h)$	if $left(\gamma, h) \downarrow \bigwedge right(\gamma, h) \downarrow$
	$=$	$right(\gamma, h) - left(\gamma, h')$	if $left(\gamma, h) \uparrow \bigwedge right(\gamma, h) \downarrow$ $h' = Min\{h' \mid h' > h \bigwedge left(\gamma, h') \downarrow\}$
	$=$	$right(\gamma, h') - left(\gamma, h)$	if $left(\gamma, h) \downarrow \bigwedge right(\gamma, h) \uparrow$ $h' = Min\{h' \mid h' > h \bigwedge right(\gamma, h') \downarrow\}$
	$=$	$\perp$	otherwise

$$
\begin{array}{lcll}
ldis(\gamma, h) & = & n - left(\gamma, h) & \text{if } left(\gamma, h) \downarrow \\
 & = & \perp & \text{otherwise} \\
 & & & \\
rdis(\gamma, h) & = & right(\gamma, h) & \text{if } right(\gamma, h) \downarrow \\
 & = & \perp & \text{otherwise}
\end{array}
$$

$$
\begin{array}{lcl}
ldis(\gamma, h) \downarrow & \stackrel{def}{=} & ldis(\gamma, h) \neq \perp \\
rdis(\gamma, h) \downarrow & \stackrel{def}{=} & rdis(\gamma, h) \neq \perp \\
ldis(\gamma, h) \uparrow & \stackrel{def}{=} & ldis(\gamma, h) = \perp \\
rdis(\gamma, h) \uparrow & \stackrel{def}{=} & rdis(\gamma, h) = \perp
\end{array}
$$

In fig.1, we illustrate *width*, *ldis* and *rdis* with examples.

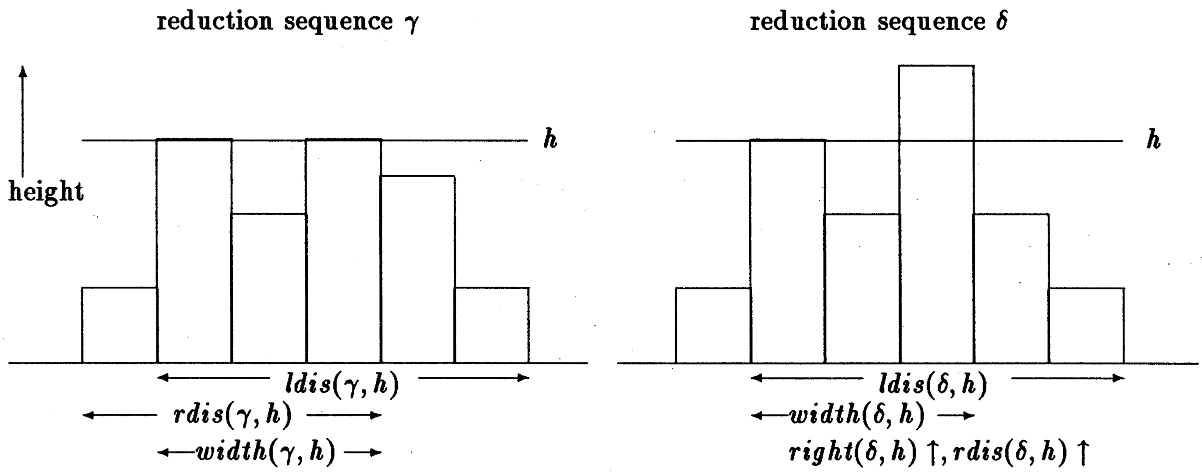

Fig.1 Definitions of ldis, rdis, width.

Example.

Let $\gamma : f(c) \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} f(g(g(c))) \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} f(g(c)) \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} f(f(g(g(c)))) \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} f(f(c)) \mathrel{\leftarrow\!\!\!+\!\!\!\rightarrow} g(c)$. Then
$left(\gamma, 1) = 0$, $left(\gamma, 2) \uparrow$, $ldis(\gamma, 1) = 5$, $ldis(\gamma, 2) \uparrow$,
$right(\gamma, 1) = 5$, $right(\gamma, 3) \uparrow$, $right(\gamma, 0) \uparrow$, $rdis(\gamma, 1) = 5$, $rdis(\gamma, 3) \uparrow$,
$width(\gamma, 1) = right(\gamma, 1) - left(\gamma, 1) = 5$, $width(\gamma, 2) = 3$, $width(\gamma, 3) = 2$, $width(\gamma, 4) = 0$

Definition of $< K(\gamma), W(\gamma) >$

$$
\begin{array}{lcl}
K(\gamma) & = & \{(h, ldis(\gamma, h)) \mid ldis(\gamma, h) \downarrow\} \\
W(\gamma) & = & \{(h, width(\gamma, h)) \mid width(\gamma, h) \downarrow\}
\end{array}
$$

Notation

We denote by $\gamma[\delta'/\delta]$ the sequence obtained from reduction sequence γ by replacing the subsequence or cut sequence δ of γ by sequence δ'.

3 Assertions

In this section, we explain how to prove that non-E-overlapping and strongly depth-preserving TRS R is CR. For this purpose, we need the following five assertions $S(k), S'(k), P(k), Q(k), Q'(k)$ for $k \geq 0$.

Assertion $S(k)$

Let $\gamma : M_0 \leftarrow\!\!\!+\!\!\!\rightarrow M_1 \leftarrow\!\!\!+\!\!\!\rightarrow \cdots \leftarrow\!\!\!+\!\!\!\rightarrow M_k$ where $|\gamma|_p = k, M_0 = \sigma(\beta), M_1 = \sigma(\alpha), M_{k-1} = \sigma'(\alpha), M_k = \sigma'(\beta)$ for some rule $\alpha \to \beta \in R$ and mappings σ, σ' and $\bar{\gamma} : M_1 \leftarrow\!\!\!+\!\!\!\rightarrow^* M_{k-1}$ is ε-invariant.

Then $\exists \delta : \sigma(\beta) \leftarrow\!\!\!+\!\!\!\rightarrow^* \sigma'(\beta)$ such that

(i) $|\delta|_p \leq k-2$

(ii) If β is a variable, then $H(\delta) < H(\gamma)$.
Otherwise, δ is ε-invariant and $H(\delta) \leq H(\gamma)$.

(iii) $\forall h \geq 0$ if $ldis(\delta, h) \downarrow$, then
$\exists h' \geq h$ such that $ldis(\gamma, h') \downarrow$ and $ldis(\delta, h) < ldis(\gamma, h')$.

Assertion $S'(k)$

Let $\gamma : M_0 \leftarrow\!\!\!+\!\!\!\rightarrow M_1 \leftarrow\!\!\!+\!\!\!\rightarrow \cdots \leftarrow\!\!\!+\!\!\!\rightarrow M_k$
where $|\gamma|_p = k, M_0 = \sigma(\beta), M_1 = \sigma(\alpha), M_{k-1} = \sigma'(\alpha), M_k = \sigma'(\beta)$ for some rule $\alpha \to \beta \in R$ and mappings σ, σ' and $\bar{\gamma} : M_1 (= \sigma(\alpha)) \leftarrow\!\!\!+\!\!\!\rightarrow^* M_{k-1} (= \sigma'(\alpha))$ is ε-invariant.

Then $\exists \delta : \sigma(\beta) \leftarrow\!\!\!+\!\!\!\rightarrow^* \sigma'(\beta)$ such that

(i) $|\delta|_p = |\gamma|_p, |\delta|_{np} \leq |\gamma|_{np} - 2$

(ii) If β is a variable, then $H(\delta) < H(\gamma)$.
Otherwise, δ is ε-invariant and $H(\delta) \leq H(\gamma)$.

(iii) $\forall h \geq 0$ if $left(\delta, h) \downarrow$, then
$\exists h' \geq h$ such that $left(\gamma, h') \downarrow$ and $left(\gamma, h') \leq left(\delta, h)$.
If $right(\delta, h) \downarrow$, then
$\exists h' \geq h$ such that $right(\gamma, h') \downarrow$ and $right(\delta, h) \leq right(\gamma, h')$.

Assertion $P(k)$

Let $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \leftarrow\!\!\!+\!\!\!\rightarrow^* M$ for some rule $\alpha \to \beta \in R$ and mapping σ where $H(\gamma) = k$ and $\bar{\gamma} : \sigma(\alpha) \leftarrow\!\!\!+\!\!\!\rightarrow^* M$ is ε-invariant.

Then, if β is not a variable, then
$\exists \delta : \sigma(\beta) \leftarrow\!\!\!+\!\!\!\rightarrow^* N \leftarrow\!\!\!+\!\!\!\rightarrow^* M$ for some N such that
$H(\delta) \leq k, M \to^* N$ and $\delta' : \sigma(\beta) \leftarrow\!\!\!+\!\!\!\rightarrow^* N$ is ε-invariant.
If β is a variable, then $\exists \delta : \sigma(\beta) \leftarrow\!\!\!+\!\!\!\rightarrow^* N \leftarrow\!\!\!+\!\!\!\rightarrow^* M$ for some N such that
$H(\delta) \leq k$, $M \to^* N$ and $H(\delta') < k$ for $\delta' : \sigma(\beta) \leftarrow\!\!\!+\!\!\!\rightarrow^* N$

Assertion $Q(k)$

Let $\gamma : M \leftarrow\!\!\!+\!\!\!\rightarrow^* N$ where $H(\gamma) \leq k$.
Then, $\exists \delta : M \leftarrow\!\!\!+\!\!\!\rightarrow^* L \leftarrow\!\!\!+\!\!\!\rightarrow^* N$ such that $H(\delta) \leq k$, $M \to^* L$ and $N \to^* L$.

Assertion $Q'(k)$

Let $\gamma_i : M \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow^* M_i$, where $H(\gamma_i) \leq k$, $1 \leq i \leq n$.
Then, $\exists \delta : M \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow^* N$ such that $H(\delta) \leq k$ and $\forall i$ $(1 \leq i \leq n)$ $M_i \rightarrow^* N$.

The assertions $S(k)$ and $S'(k)$ are similar to the Elimination lemma in [7]. For any reduction sequence $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow^* \sigma'(\alpha) \rightarrow \sigma'(\beta)$ for some rule $\alpha \rightarrow \beta$ and mappings σ, σ' where $\bar{\gamma} : \sigma(\alpha) \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow^* \sigma'(\alpha)$ is ε-invariant, $S(k)$ ensures that there exists $\delta : \sigma(\beta) \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow^* \sigma'(\beta)$ such that $|\delta|_p \leq |\gamma|_p - 2$, $H(\delta) \leq H(\gamma)$ (where δ is ε-invariant or $H(\delta) < H(\gamma)$) and $K(\delta) \ll K(\gamma)$. Here, $\ll$ is the multiset ordering of a lexicographic ordering $<$. And $S'(k)$ ensures that there exists $\delta' : \sigma(\beta) \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow^* \sigma'(\beta)$ such that $|\delta|_p = |\gamma|_p$, $|\delta|_{np} \leq |\gamma|_{np} - 2$, $H(\delta) \leq H(\gamma)$ (where δ is ε-invariant or $H(\delta) < H(\gamma)$) and $W(\delta) \lesseqqgtr W(\gamma)$. Here, $\lesseqqgtr$ is $\ll$ or $=$.

To prove these assertions, we use the following properties for $left, right, width$.

Property 1

Let $\gamma : M_0 \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow M_1 \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow \cdots \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow M_k$,
$\delta : N_0 \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow N_1 \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow \cdots \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow N_k$.

1. Assume that for $h > 0$, $left(\delta, h) \downarrow$ and there exists j such that $j \leq left(\delta, h)$ and $h(M_j) \geq h$.
   Then, there exists $h' \geq h$ such that $left(\gamma, h') \downarrow$ and $left(\gamma, h') \leq left(\delta, h)$.
2. Assume that for $h > 0$, $right(\delta, h) \downarrow$ and there exists j such that $right(\delta, h) \leq j$ and $h(M_j) \geq h$.
   Then, there exists $h' \geq h$ such that $right(\gamma, h') \downarrow$ and $right(\gamma, h') \geq right(\delta, h)$.

Property 2

If $H(\gamma) > H(\delta)$, then $K(\gamma) \gg K(\delta)$ and $W(\gamma) \gg W(\delta)$.
Here, $\gg$ is the multiset ordering of a lexicographic ordering $>$.

These proofs are obvious by the definitions of left, right and width, etc.

We first prove $S(k)$ and $S'(k)$ by induction on $k \geq 0$, where k is the number of parallel reduction steps of γ. In the case of $k > 2$, we prove $S(k)$ and $S'(k)$ by induction on $weight(\gamma)$ which is defined as follows:

$$weight(\gamma) \;=\; \sum_{\gamma_i \in \Gamma} |\gamma_i|_{np}$$

where $\Gamma = \{\gamma_i \mid \gamma_i = \bar{\gamma}/u_i \text{ for some } u_i \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)\}$,
$\bar{\gamma} : \sigma(\alpha) \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow^* \sigma'(\alpha)$.

1. Basis, i.e., the case of $weight(\gamma) = 0$
   The proof is straightforward.
2. Induction step, i.e., the case of $weight(\gamma) > 0$
   Let $\gamma_1 = \bar{\gamma}/u_1 : L_1 \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow L_2 \cdots \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow L_{k-1}$ where $\gamma_1 \in \Gamma$ and $L_i = M_i/u_1$, $1 \leq i \leq k-1$.
   Then, there exist i, j such that $1 \leq i < j < k-1$ and
   $\delta_1 : L_i \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow L_{i+1} \cdots \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow L_j \leftrightarrow\!\!\!\!+\!\!\!\!\rightarrow L_{j+1}$
   where $L_i = \theta(\beta')$, $L_{i+1} = \theta(\alpha')$, $L_j = \theta'(\alpha')$, $L_{j+1} = \theta'(\beta')$ for some rule $\alpha' \rightarrow \beta'$ and mappings θ, θ'.

By the induction hypothesis $S(k')$, where $k' = |\delta_1|_p$, there exists $\eta_1 : L_i \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow^* L_{j+1}$ satisfying the conditions (i), (ii) and (iii). Let $\eta_1' = ((L_i \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow L_i \cdots \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow L_i); \eta_1)$ where $|\eta_1'|_p = |\delta_1|_p$.
Let $\gamma' = \gamma[\eta_1'/\delta_1]$. Then, obviously $weight(\gamma) > weight(\gamma')$ holds. Hence, by the induction hypothesis that $S(k)$ holds for γ', it follows that $S(k)$ holds for γ.

The proof of $S'(k)$ is similar to that of $S(k)$.

We then prove that $Q(k) \Rightarrow Q'(k)$ for all $k \geq 0$. Using these results, we can prove $P(k) \bigwedge Q(k)$ by induction on $k \geq 0$.

Outline of the proof of $P(k) \bigwedge Q(k)$.
We first prove $P(k)$. Basis: $k = 0$. The proof is obvious.
Induction step: Let $\gamma : M_0 \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow M_1 \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow M_2 \cdots \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow M_n$ where $H(\gamma) = k$, $M_0 = \sigma(\beta)$, $M_1 = \sigma(\alpha)$ and $M_n = M$. Let $\bar{\gamma} : M_1 \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow M_2 \cdots \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow M_n$. We prove $P(k)$ by induction on the following $weight(\gamma)$.

$$weight(\gamma) \quad = \quad \bigsqcup_{\gamma_i \in \Gamma} K(net(\gamma_i^R))$$

where $\Gamma = \{\gamma_i \mid \gamma_i = \bar{\gamma}/u_i \text{ for some } u_i \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)\}$.
Here, γ_i^R is the reverse sequence of γ_i.
Note that if $\Gamma = \phi$, then $weight(\gamma) = \phi$.

1. Basis: the case of $weight(\gamma) = \phi$, i.e., all the reductions of γ occur in the variable parts of $\sigma(\alpha)$.

   We can prove $P(k)$ by using the induction hypothesis $Q(k-1)$ and the strongly depth-preserving property.

2. Induction step: the case of $weight(\gamma) \gg \phi$ i.e., some reduction occurs in the non variable part.

   By the definition of γ_1^R, then there exists an ε-reduction.
   Let $\delta = net(\gamma_1^R) : (L_0 \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow L_1 \cdots \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow L_m)$ where $m \leq n, L_0 = M_n/u_1, L_m = M_1/u_1$.
   There are two cases depending on whether there exists
   $\xi : L_i(= \sigma'(\beta')) \leftarrow^{\varepsilon} L_{i+1}(= \sigma'(\alpha')) \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow^* L_j(= \sigma''(\alpha')) \rightarrow^{\varepsilon} L_{j+1}(= \sigma''(\beta'))$
   for some i, j $(1 \leq i < j < m)$, where $L_{i+1} \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow^* L_j$ is ε-invariant.

   (a) The case in which δ includes such ξ.
   By $S(|\xi|_p)$, there exists $\xi' : L_i \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow^* L_{j+1}$ satisfying the conditions (i), (ii), (iii).
   Let $\delta' = \delta[\xi'/\xi]$ and $\gamma' = \gamma[\gamma_1'/\gamma_1]$ where $net(\gamma_1'^R) = \delta'$ and $net(\gamma_1^R) = \delta$.
   By $weight(\gamma) \gg weight(\gamma')$, the induction hypothesis for γ' ensures that $P(k)$ holds for γ.

   (b) The case in which δ does not include such ξ.
   In this case, δ includes ε-reductions, but the direction of the ε-reductions is left-to-right by the non-E-overlapping property.
   Using a finite number of the induction hypothesis $P(k'), k' < k$, we can prove that there exists $\eta : L_0 \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow^* N \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow^* L_i$ for some term N and i $(0 < i \leq m)$ such that $H(\eta) \leq H(\delta), L_0 \rightarrow^* N$ and either $i = m$ and $\eta' : N \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow^* L_i$ is ε-invariant or $H(\eta') < H(\delta_i)$ holds where $\eta' : N \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow^* L_i$ and $\delta_i : L_0 \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow L_1 \cdots \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow L_i$.

Let $\bar{\delta} = \delta[\eta'/\delta_i]$. Then, $\bar{\delta}$ is ε-invariant or $K(\delta) \gg K(\bar{\delta})$ holds. Let $\gamma' = \gamma[\gamma_1'/\gamma_1]$ where $\bar{\delta} = net(\gamma_1'^R)$ and $\delta = net(\gamma_1^R)$. Then, $weight(\gamma) \gg weight(\gamma')$ holds, so that the induction hypothesis $P(k)$ for γ' ensures that $P(k)$ holds for γ.

Next, we prove $Q(k)$ by induction on $(H(\gamma), W(\gamma), \varepsilon(\gamma))$, where $\varepsilon(\gamma)$ is the number of ε-reductions in γ and $W(\gamma) = \{(h, width(\gamma, h)) \mid width(\gamma, h) \downarrow\}$.

If $H(\gamma) \leq k-1$ or γ has no ε-reductions, then the proof can be reduced to that of $Q(k-1)$. So, let $H(\gamma) = k$ and γ has ε-reductions.

There are two cases depending on whether there exists a subsequence
$\gamma_1 : N_1 \leftarrow^{\varepsilon} N_2 \leftarrow\!\!\!+\!\!\!\rightarrow^* N_3 \rightarrow^{\varepsilon} N_4$
of γ for some $N_i, 1 \leq i \leq 4$, where $N_2 \leftarrow\!\!\!+\!\!\!\rightarrow^* N_3$ is ε-invariant.

1. The case in which γ includes such γ_1.
   In this case, we apply $S(|\gamma_1|_p)$ or $S'(|\gamma_1|_p)$ and obtain $\delta_1 : N_1 \leftarrow\!\!\!+\!\!\!\rightarrow^* N_4$ satisfying the conditions (i),(ii) and (iii).

   Let $\gamma' = \gamma[\delta_1/\gamma_1]$. Then, either $W(\gamma) \gg W(\gamma')$ or $W(\gamma) = W(\gamma')$ and δ_1 is ε-invariant. In either case, the induction hypothesis for γ' ensures that $Q(k)$ holds for γ.

2. The case in which γ does not include such γ_1.
   We can prove this case by using $P(k)$ and $Q(k-1)$. But, the details are omitted.

Since $Q(k), k > 0$, ensures that TRS R is CR, we have the following our main theorem.

Main Theorem
A TRS R is CR if R is non-E-overlapping and strongly depth-preserving.

Matsuura et al.[6] showed that if a TRS R is non-ω-overlapping and depth-preserving, then R is non-E-overlapping, so that we have the following corollary.

Corollary
A TRS R is CR if R is non-ω-overlapping and strongly depth-preserving.

Note
Whether R is non-ω-overlapping or not can be checked efficiently.

4 Concluding Remarks

In this paper, we have shown that there exists a non-E-overlapping and depth-preserving TRS which is not CR, but all the non-E-overlapping and strongly depth-preserving TRS's satisfy the CR property.

Finally, we make a comment on the strongly depth-preserving property. This property is defined by the depth-preserving property and the condition that for each rule $\alpha \rightarrow \beta$ and for any $x \in V(\alpha)$, all the depths of the x occurrences in α are the same. By replacing the restriction on α by that on β, we can define an analogous property. That is, this new property is defined by the depth-preserving property and the condition that for each rule $\alpha \rightarrow \beta$ and for any $x \in V(\beta)$, all the depths of the x occurrences in β are the same. However, this new property and non-E-overlapping do not necessarily ensure CR. For example, TRS $R_6 = \{f(g(x), x) \rightarrow a,\ c \rightarrow h(c, g(c)),\ h(x, g(x)) \rightarrow f(g(x), h(x, g(c)))\}$ is non-E-overlapping and satisfies this new condition, but R_6 is not CR.

It will be a next step following the work of this paper to study the CR property of E-overlapping and strongly depth-preserving TRS, that is, to find restriction conditions that E-critical pairs must satisfy for ensuring the CR property of strongly depth-preserving TRS's.

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